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Loop models for conformal field theories

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Abstract
By interpreting the fusion matrix as an adjacency matrix we associate a loop model with every primary operator of a generic conformal field theory. The weight of these loop models is given by the quantum dimension of the corresponding primary operator. Using the known results for the $O(n)$ models, we establish a relationship between these models and SLEs. The method is applied to WZW, $c < 1$ minimal conformal field theories and other coset models.

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1. Introduction

The study of statistical mechanics systems related to loop models is interesting both from the physical and the mathematical points of view. Most of the statistical models studied in physics, from the Ising and the q-state Potts model to more complex vertex models, can be represented in terms of loops. The loop representation of the Ising model is very easy to understand: loops correspond to domain walls separating regions of opposite magnetization. If the Ising model is defined on the infinite plane or in a box with periodic boundary conditions, then the boundaries of domain walls do not have open ends or branch points. The same property is true in the more general Potts model if we define domain walls appropriately.

A fundamental model, the starting point in the construction of more complicated loop systems, is the $O(n)$ model, and its connection with exactly solvable models led to the introduction of powerful integrable model technologies. A first interesting progress in this direction was the discovery of a critical point in the $O(n)$ model made by Nienhuis [1], however the rigorous proof of this result is still missing.

After the advent of conformal field theory (CFT) as a tool to classify two-dimensional critical phenomena, the main efforts to understand critical loop models have been based on the CFT approach. Many critical loop models were also defined and studied with other methods, such as the Yang–Baxter equation [2] and non-rigorous Coulomb gas methods [3]. For a review of the Coulomb gas method for more complex loop models, see [4]. The critical points of these models are described by CFTs, the early versions correspond to well-known CFTs such as the $c < 1$ minimal models, but most of the recent proposals link to CFTs with extra symmetries like the Wess–Zumino–Witten (WZW) models [5]. For recent progress in this direction, see [6, 7]. In parallel with the progress on CFTs as a purely algebraic method to classify critical points of statistical systems, Pasquier [9] proposed the ADE lattice models, a generalization of the restricted solid-on-solid models of Andrews *et al* [8], as the physical candidates for particular CFTs. Pasquier's work is closely related to the Cappelli–Itzykson–Zuber ADE classification of CFTs [10] established by noting that the operator content of many CFTs is constrained by the modular invariance of the partition function on torus geometry and is related to Coxeter diagrams.

The definition of the corresponding loop models and the study of the boundary changing operators for the A series were initiated by Saleur and Bauer [11]. The A series is related to dense critical loop models, while the dilute versions were studied later by Kostov [12].

Some progress has been made in the study of loops related to the remaining ADE lattice systems, but many of their properties are still unknown and in particular the general fused ADE models are not well explored. There are at least a couple of reasons that make the study of loop models related to statistical systems interesting. First, from the mathematical point of view it may give us good candidates for the Schramm-Loewner evolution (SLE), a method discovered by Schramm [13] to classify conformally invariant curves connecting two distinct boundary points in a simply connected domain. The parameter describing the curve is the drift parameter κ. (For recent reviews, see [14, 15].) The exact relation between SLEs and CFTs was discovered by Bauer and Bernard [16]: using the method of null vectors of CFT they were able to find a simple relation between the CFT central charge and the drift κ. This method, based on the original results of Schramm, was generalized later to loops without open ends by Sheffield and Werner [17] and named conformal loop ensemble (CLE). The connection between ADE loops and CLE was studied by Cardy [18] in both dilute and dense cases by mapping the height variable of ADE lattice models to the $O(n)$ model.

Another reason that makes this study interesting is related to the study of topological quantum field theory in 2 + 1 dimensions and to its application to topological quantum computation [19]. The ground-state wavefunction of these topological models coincides with the loop ensemble. This means that the ground-state correlators in (2 + 1)-dimensional topological quantum field theories are equal to particular correlators in 2D classical loop models [20]. The corresponding loops have weights related to the quantum dimension of the operators appearing in the model. It seems that mathematically one can classify the ground state of topological theories by two-dimensional critical loop models or equivalently by CFTs.

The above considerations suggest that a classification of the loop representations related to CFTs may be relevant for a wide range of applications.

In the following, using the methods introduced by Cardy [18], we shall define a loop representation for a generic CFT. To define these loop models we use the fusion matrix of CFTs as an adjacency matrix and map the resulting height model on a $O(n)$ system. We find a different $O(n)$ model for every CFT primary operator and observe that the weights of the loops coincide with the quantum dimensions of the corresponding primary operator. Using the link between $O(n)$ and CLEs one can map every CFT to SLEs and to CLEs.

This allows the investigation of the link between SLEs and CFTs to be carried out without the use of, sometimes quite complicated, the null vector method. The main ingredient of the method is the S matrix or the complete form of the fusion matrix which is complex for some extended CFTs.

The paper is organized as follows: in section 2 we use the fusion matrix as an adjacency matrix and define a height model using the S matrix as a weight for the plaquettes in a triangular lattice.

Following [18], using the height model we will associate with every height configuration an ensemble of loops, and by the Verlinde formula find that the partition function of the loop ensemble matches that of a particular $O(n)$ model.

In section 3 we investigate the loop representation of some simple CFTs such as the Ising and tricritical Ising models, the $c < 1$ minimal models, the WZW and in particular the $SU_k(2)$ models. Our prediction for the SLE drift parameter κ is in agreement with all the previously known examples.

In section 4 we will generalize the methods introduced in section 2 and define loop models on the square lattice related to the dense phase of the $O(n)$ models. The definition of loops on the square lattice is more natural because the models arising from our definition are always critical without the need of a fine-tuning of the parameters. In section 5 we will briefly discuss the connection between these results and boundary CFT and show that the models are well defined also in the presence of a boundary.

Section 6 contains our conclusions with a brief description of the work in progress motivated by these results.

2. Loop models for general CFTs

To define loop models for a given CFT, we shall use the modular S matrix. To define the S matrix we first need the expression of the modular invariant partition function on the torus geometry in terms of the characters and the parameter $\tau = \mathrm{e}^{\frac{2\pi i\beta}{R}}$, where R is the space length of the system and β is the time period, as follows:

$$Z = \sum M_{h,\bar{h}} \chi_h(\tau) \chi_{\bar{h}}(\overline{\tau}). \tag{2.1}$$

In (2.1) the sum is over the operator content of the theory and $M_{h,\bar{h}}$ is the non-negative integer that specifies how many times a given representation enters, that is the number of primary operators with scaling dimensions $(h, \bar{h})$. The function $\chi_h = \mathrm{tr}\left(\tau^{L_0 - \frac{c}{24}}\right)$ is the character of the theory with the sum over the descendants of the primary field with conformal weight h.

From now on we shall use a, b, c to label the primary operators (they are not the weights of primary operators) in the theory independent of the holomorphicity of the operator. Then one can define the modular S matrix by the modular transformation which does not change the partition function but acts nontrivially on the characters

$$\chi_a\left(-\frac{1}{\tau}\right) = \sum_b S_a^b \chi_b(\tau). \tag{2.2}$$

Another important ingredient is the fusion rule related to the operator product expansion of a pair of primary operators,

$$\phi_a \cdot \phi_b = \sum_c N_{ab}^c \phi_c. \tag{2.3}$$

In (2.3) N_{ab}^c is the fusion coefficient and the rhs involves the entire tower of descendants of the primary operator ϕ_c. It is worthwhile to mention that it is possible to interpret the fusion coefficients as the elements of a matrix $(N_a)_b^c = N_{ab}^c$, so we have a fusion matrix for every primary operator.

To define a loop model, we interpret the fusion matrix as the adjacency matrix associated with a graph [18]. The graph of a primary operator ϕ_a has g vertices, where g is the number of

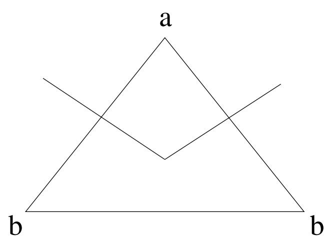

Figure 1. A triangular plaquette with $a \neq b$ and the corresponding curve segment on the dual honeycomb lattice.

primary operators in the theory and edges connecting pairs of vertices (b, c) when $N_{ab}^{c} = 1$. Following [18] one can define a height model on the triangular lattice by imposing that the height h_j at the site j can take values $0, 1, \ldots, g-1$. Then constrain the heights at neighboring sites according to the incidence matrix associated with a given primary field ϕ_a: only neighbor heights h_k and h_j with $(N_a)_{h_j}^{h_k} = 1$ are admissible. One should note that according to the fusion matrix rule neighbor heights may be identical, which is not the case for ADE models. Actually, for a consistent definition of loop models on a triangular lattice at least two of the heights at the corners of an elementary triangular plaquette should be equal[1], then the weights for the elementary plaquette are defined as follows. If the heights of the plaquette are (c, b, b) with $c \neq b$ then the weight is $x\left(\frac{S_k^b}{S_k^c}\right)^{1/6}$, where S is the modular matrix and k is arbitrary. If the heights are all equal then the weight is 1 except for those with $N_{ab}^{b} \neq 0$ that have weights 1 or x depending on the particular model considered, as will be explained below.

The next step is to mark triangles with unequal heights (c, b, b) drawing a curved segment on the dual honeycomb lattice [18] and linking to the center the midpoints of the two edges with different heights (b and c) at the extremes (see figure 1). The difference with the more standard ADE models is related to the equal height (b, b, b) plaquette with $N_{ab}^{b} = 1$. For plaquettes with height (b, b, b) there are three possibilities for drawing curve segments. In these cases we will suppose, as in percolation problems, that the lines in the dual honeycomb lattice choose randomly two of the edges of triangular lattice consistently with the other sites. This defines a loop configuration for every height configuration. Summing over the admissible values of heights consistent with a given loop configuration we find

$$\sum_b (N_a)_b^c \frac{S_k^b}{S_k^c} = \frac{S_k^a}{S_0^k}, \tag{2.4}$$

where the sum is just over b. To get this formula we have used the Verlinde formula for CFTs, $\sum_b (N_a)_b^c \frac{S_k^b}{S_k^0} = \frac{S_k^a}{S_0^k}\frac{S_k^c}{S_k^0}$, which means that the bth element of the eigenvector of N_a with eigenvalue $\frac{S_k^a}{S_0^k}$ is given by $\frac{S_k^b}{S_k^0}$. We always take $k = 0$ to get the largest eigenvalue and have positive real weights in our height models. For properly defined fusion matrices these eigenvalues will coincide with the quantum dimension of ϕ_a:

$$d_a = \frac{S_0^a}{S_0^0}. \tag{2.5}$$

[1] We can have loops in our graph but at least for minimal models and restricting the analysis to primary operators of interest, it is possible to prove that there are no graphs with loops smaller than or equal to three. In general, we just define models for graphs with loops longer than three or even without loops.

Summing iteratively over all heights of all clusters gives a factor d_a for each closed loop, so the partition function of our model has a $O(n)$-like partition function

$$Z = \sum x^l d_a^N, \tag{2.6}$$

where l is the number of bonds in the loop configuration and N is the number of loops.

Using this method, we are then able to associate with every primary field of a given CFT an $O(n)$ model with $n = d_a$. Let us discuss some simple but general properties of these models. First, it is not difficult to see that for the identity operator our incidence graph has g disconnected points with a blob attached to them to indicate that they are adjacent to themselves. This means that the configuration space is decomposable and that all the heights should be equal. In loop language means that we have a percolation or an Ising loop model depending on the value of x. It is not difficult to see that for any non-connected graph our height configuration space is decomposable and that we can treat different parts of our graph separately. Another interesting fact is related to the consistency of different parts of the graph: if we write for different connected components of the graph the corresponding adjacency matrices then all the parts will have the same loop weights and the same quantum dimension. Therefore, we can consistently keep only one part of the graph and find the loop model associated with it. We shall clarify further this issue later by discussing some minimal model examples in details.

It was found long time ago in [1] that the $O(n)$ model possess a dilute critical point for $n \leqslant 2$ with $x_c = \frac{1}{\sqrt{2+\sqrt{2-n}}}$: correspondingly our loop models will have a critical point just for the fields with quantum dimension smaller than 2. The $O(n)$ model has another critical regime, the so-called dense phase, for $x = (x_c, \infty)$ which corresponds to a different universality class. The mapping to the $O(n)$ model helps us to find the connection with CLE. From Coulomb gas arguments we know that, in the dilute regime, the loop weight has the following relation with the drift in the CLE equation:

$$d_a = -2\cos\left(\frac{4\pi}{\kappa}\right). \tag{2.7}$$

We will use this equation to establish a connection between our models and CLEs and find that the prediction is in excellent agreement with previous results. We should mention that for the dense phase the above equation is still true if we work in the region $4 \leqslant \kappa \leqslant 8$. In the following section we will investigate the loop representation of some simple CFTs.

3. Some examples

In this section we show how the method described in the previous section is working for some simple models such as minimal models, $SU_k(N)$ WZW models and coset models.

3.1. Minimal models

Consider the minimal models $M(p', p)$, the simplest case is the Ising model $M(4, 3)$. This model has three primary operators [5]: $1, \sigma, \epsilon$ with graphs depicted in figure 2. We discussed in the previous section how the operator 1 is related to percolation or the Ising model with $\kappa = 6$ or 3. The other two cases are more interesting, for example, the graph of N_ϵ has a part similar to the A_2 graph and a disconnected point, using $d_\epsilon = -2\cos\left(\pi\frac{4}{3}\right) = 1$, it is not difficult to see that both parts have the same dominant eigenvalues corresponding to $\kappa = 3$ which is in agreement with [18]. The graph for N_σ is similar to A_3 and gives us $\kappa = \frac{16}{3}$ in

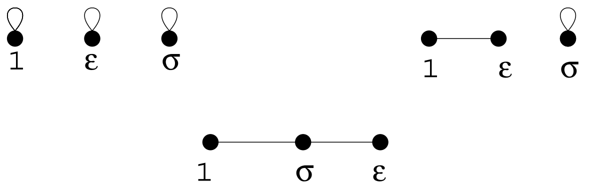

Figure 2. Graphs of fusion matrices of primary operators in the Ising model, from left to right the fusion graph of 1, ϵ and σ.

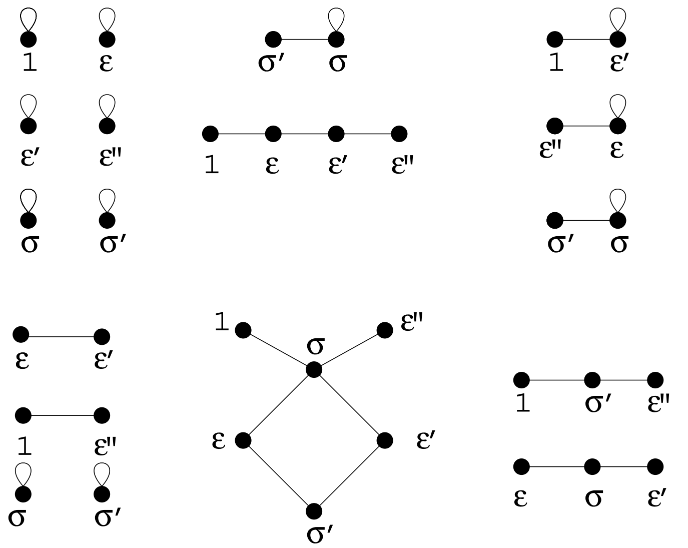

Figure 3. Graphs of fusion matrices of primary operators in three critical Ising models: in the upper row from left to right the fusion graph of 1, ϵ and ϵ', in the lower row from left to right the fusion graph of ϵ'', σ and σ'.

the dense phase, which is the dual of the previous case. It is quite surprising that the critical behavior of these loops is completely in agreement with the known Ising model.

The next simple minimal model is the tricritical Ising model $M(5,4)$ with 6 primary operators and 15 different fusion relations [5]. The fusion matrix graphs for this model are shown in figure 3. The familiar cases with the critical loop models are $1, \sigma', \epsilon$. The graph of $N_{\sigma'}$ has two A_3 parts, using $d_{\sigma'} = -2\cos\left(\pi\frac{5}{4}\right)$ it is easy to get $\kappa = \frac{16}{5}$ which is again in agreement with [18]. The case N_ϵ is similar to A_4 and related to $\kappa = 5$. One can check simply that the other part of the graph has the same eigenvalue and so related to the same kind of loop model, this kind of graphs are the first kind of non-familiar height models corresponding to loop models. We also note that one part of this graph is similar to the second level of the fusion of the A_4 model. We have two more graphs related to critical loop models, $N_{\epsilon'}$ has quantum dimension equal to N_ϵ so they have the same properties. The quantum dimension related to $N_{\epsilon''}$ is equal to 1 and so again we have percolation or the Ising model. The last one is the N_σ which has quantum dimension bigger than two and so we cannot associate a critical loop model with this primary operator, we note that this graph is also similar to the ground-state adjacency diagram of the second-level fusion of the A_6 model [6].

Let us summarize the meaning of disconnectedness in the graphs of different fusion matrices of primary operators. For both of the above models if we treat different parts of the graph as the different models then we can finally find the same loop models. This means that the different blocks of our block diagonalizable adjacency matrices have the same largest eigenvalues and here our definition of the loop model is consistent. One can always choose one block and get the corresponding loop model. This is exactly the same as the definition of the well-known ADE models, for these models always getting one block of our fusion matrix is enough. The degeneracy in the eigenvalues of the fusion matrix is the natural result that comes from the most general property of the fusion matrix: different fusion matrices commute with themselves. It seems that the above argument is a general property of many different CFTs especially the more general cases which we will investigate in the following.

The above calculation is tractable in the general case of minimal models $M(p', p)$ with Kac formula for the conformal weights of primary operators

$$h_{r,s} = \frac{(p'r - ps)^2 - (p - p')^2}{4pp'}, \tag{3.1}$$

where (r, s) is the label of primary operators in the Kac table and $1 \leqslant r \leqslant p-1$, $1 \leqslant s \leqslant p'-1$. For the known primary operators in the non-extended Kac table, it is not difficult to see that because of $(r, s) = (p - r, p' - s)$ there is a huge redundancy in this rectangle. The S matrix for minimal models has the following form:

$$S_{rs,\rho\sigma} = 2\sqrt{\frac{2}{pp'}}(-1)^{1+s\rho+r\sigma} \sin\left(\pi \frac{p}{p'} r\rho\right) \sin\left(\pi \frac{p'}{p} s\sigma\right). \tag{3.2}$$

Using equation (2.5) the quantum dimension of the primary operator ϕ_{rs} has the following form:

$$d_{rs} = (-1)^{r+s} \frac{\sin\left(\pi \frac{p}{p'} r\right) \sin\left(\pi \frac{p'}{p} s\right)}{\sin\left(\pi \frac{p}{p'}\right) \sin\left(\pi \frac{p'}{p}\right)}. \tag{3.3}$$

We should mention that the above equation respects the symmetry $(r, s) = (p - r, p' - s)$ for the unitary minimal models. One can investigate the properties of graphs for different operators but the most interesting cases are $(r, s) = (1, 2)$ and $(r, s) = (2, 1)$. The fusion matrices for these cases are the following symmetric matrices:

$$N^{(m,n)}_{(1,2)(r,s)} = \delta_{m,r}(\delta_{n,s+1} + \delta_{n,s-1}), \qquad N^{(m,n)}_{(2,1)(r,s)} = \delta_{n,s}(\delta_{m,r+1} + \delta_{m,r-1}). \tag{3.4}$$

It is not difficult to see that the graph for these two cases do not have any loop and so we do not need to worry about the possibility of the definition of the loop model. For example, one can show that the graph of $(r, s) = (1, 2)$ for unitary minimal models, $M(p + 1, p)$, has always a part similar to the graph A_{p-1} and $(r, s) = (2, 1)$ has a part similar to A_p. The quantum dimension of $(r, s) = (1, 2)$ is

$$d_{12} = -2\cos\left(\pi \frac{p'}{p}\right). \tag{3.5}$$

If we suppose that the loop model is at the critical regime then it is easy to see that this loop is related to CLE with the following drift:

$$\kappa = 4\frac{p}{p'}, \tag{3.6}$$

which is exactly the same as the κ predicted for minimal models by using null vectors [16]. One can see this by rewriting the level 2 null vector of minimal models as follows:

$$L_{-2} - \frac{3}{2(2h + 1)} L_{-1}^2 = L_{-2} - \frac{p}{p'} L_{-1}^2, \tag{3.7}$$

for the unitary Virasoro minimal models and comparing it with the null-vector-like relation in SLE [16]. It seems that there is a close relation between this loop model for $(r, s) = (1, 2)$ and CLE. The same calculation is possible for $(r, s) = (2, 1)$ and the result is $d_{21} = -2\cos\left(\pi \frac{p}{p'}\right)$ with $\kappa = 4\frac{p'}{p}$ which is apparently the dual of the previous case and consistent with CLE prediction[2]. In this construction it is also possible to get other CLE drifts by considering the dense (dilute) phase of the loop model corresponding to $\phi_{1,2}$ ($\phi_{2,1}$). In this level it is not clear how one can choose the right phase but the algorithm for minimal models as we noted is taking dilute phase for $\phi_{1,2}$ and dense phase for $\phi_{2,1}$. For exact correspondence one needs to investigate the problem in the partition function level.

These two cases are the familiar cases but as we noted in the tricritical Ising model these are not the only loop models with critical point; in fact, it is possible to have many critical loop models for a minimal model with generic p and p' just we need to have quantum dimension smaller than or equal to two. Appearing the A_{p-1} and $A_{p'-1}$ graphs in our fusion matrix is not just an accident because in fact these are the graphs that already appeared in the ADE classification of minimal models [10]. From the other point we know that these graphs have lattice statistical physics counterparts with the loop representations investigated in [18], so appearing a close connection between the classification of modular invariant CFTs in two dimensions with respect to the ADE graphs with our loop representation is not so much surprising. At least for the minimal models this connection is obvious and the fusion graphs of some special operators are exactly the same as the ADE diagram of the corresponding CFT.

For later purposes let us write the results explicitly for the case $M(6, 5)$ which is describing the three-state Potts model, using equation (3.3) we find the following quantum dimensions:

$$\begin{aligned} &d_{(1,1)} = 1, && d_{(1,2)} = 2\cos\left(\frac{\pi}{5}\right), && d_{(1,3)} = 2\cos\left(\frac{\pi}{5}\right), && d_{(1,4)} = 1, \\ &d_{(2,1)} = \sqrt{3}, && d_{(3,1)} = 2, && d_{(2,2)} = 2\sqrt{3}\cos\left(\frac{\pi}{5}\right), && d_{(2,3)} = 2\sqrt{3}\cos\left(\frac{\pi}{5}\right), \\ &d_{(3,3)} = 4\cos\left(\frac{\pi}{5}\right), && d_{(3,2)} = 4\cos\left(\frac{\pi}{5}\right), && d_{(2,4)} = \sqrt{3}, && d_{(3,4)} = -2. \end{aligned} \tag{3.8}$$

It is evident that different primary fields can have the same quantum dimension and so the same loop representations. For example, $\kappa_{1,2} = \kappa_{1,3} = \frac{10}{3}$ and $\kappa_{2,1} = \kappa_{2,4} = \frac{24}{5}$, however we know from the connection between CFT and SLE that the fields $(1, 2)$ and $(2, 1)$ are connected to SLE by null vector so we need to answer this question how we can choose the right operator. For the three-state Potts model, we can answer the question by looking at the modular invariancy of the model which is responsible for constraining the operator content of the model to just six operators, $(1, 1), (2, 1), (3, 1), (4, 1), (3, 3), (1, 3)$ in the spin representation of the model [5]. However, for the Fortuin–Kasteleyn representation of the model this operator content is not enough and we should insert disorder operators as well. The theory has extra W symmetry which is responsible in decreasing the primary operators of the theory from ten to six. If we look at the model with the above symmetry then we can show that this model is not just a sub-theory of the minimal model $M(6, 5)$, the fields in this model represent different primary fields with different characters, however the conformal dimensions are equal. Using operators $W(z)$ and $\overline{W}(\bar{z})$, with weight equal to three, one can show that $\phi_{(1,3)} = W_{-1}\overline{W_{-1}}\phi_{(1,2)}, \phi_{(2,3)} = W_{-1/2}\phi_{(2,2)}$ and $\phi_{(4,1)} = W_{-1/2}\phi_{(2,1)}$, where W_n and $\overline{W}_n, n = 0, \pm1, \pm2, \ldots,$ are the mode operators. The above relations show that the

[2] The outer boundary of fractal curve with $\kappa > 4$ is given by another fractal curve with $\kappa := 16/\kappa$ named the dual curve. This curve has the same central charge as the original one.

operators with the same quantum dimension are in fact related with the W symmetry of the model. In other words, they are appearing in the modular partition function together [5], so the equality of quantum dimension of different fields is assigned for the internal extra symmetry of the model. This interpretation of the three-state Potts model is also consistent with the parafermionic interpretation [21]. The same story is true for the tricritical Ising model which has also supersymmetry at the critical point. This symmetry is responsible for the equality of quantum dimension of the operators $\mathbf{I}$ and ϵ'' and also the operators ϵ and ϵ'. Both couples come from Neveu–Schwarz sector of superconformal symmetry of the model [5] and one is the descendent of the other in the presence of supersymmetry. The other important thing to mention is to get the connection to the physical system we need to investigate the symmetries of the statistical model specially those that are connected to the symmetries of the domain walls. For example, for the three-state Potts model the fluctuating domain walls should respect the Z_3 symmetry. We know from the boundary conformal operators that the corresponding responsible operator is the spin operator [22], but free boundary condition is not respecting Z_3 symmetry and so in the modular invariant partition function we do not have the corresponding operator, so we always need to have some information coming from the boundary CFT as well.

The above calculation can be generalized for the extended Kac table too, for example take the simple case $M(3,2)$ which is related to CFT with zero central charge. The loop model for the operator $(1,2)$ has $\kappa = \frac{8}{3}$ which is self-avoiding random walk with zero quantum dimension. The loop model of $(2,1)$ has $\kappa = 6$ which is percolation. These two cases are again consistent with the known results. We should mention that the above naive argument is just coming from equation (3.6) and does not mean that the adjacency matrix is just A_1. The other simple case is $M(2,1)$ with quantum dimension -2. In this case we have $\kappa = 2$ and $\kappa = 8$, corresponding to loop-erased random walk and spanning trees, respectively. One can simply investigate the extended Kac table of more complex CFTs like the Ising model and find some new loop models with zero or negative quantum dimension which is related to $O(n)$ models. For example, the Ising model $(1,4)$ has zero quantum dimension and $(1,5)$ has negative quantum dimension. The zero and negative quantum dimensions cannot be related to unitary CFTs and as it is quite well known the extended Kac table models are not unitary, they are related to logarithmic CFTs (LCFT) with special kinds of fusion matrices. For example, let us briefly see the most familiar LCFT, $c = -2$. For this case if we just suppose the fields with $(r,s) = (1,1), (1,2), (1,3), (1,4), (1,5)$ then the fusions of the operators make a closed algebra [23]. Using the fusion matrices of operator $(1,2)$, which has also integer elements more than one, will give us the same dominant eigenvalue equal to zero, this is the same as our above naive argument. We think that these loop models are in close relation with the lattice logarithmic minimal models introduced by Pearce *et al* [24].

3.2. $SU_k(N)$

Let us investigate the CFTs with additional symmetries specially WZW models and for simplicity the most simple and familiar case, i.e., $SU_k(2)$. The fusion matrix of these models are well known; for example, see [5], by generating loop model as before we will have some loops with weights equal to the quantum dimension of the corresponding primary operators, for this case quantum dimensions are completely familiar and has the following form:

$$d_a = \frac{\sin(\pi(a+1)/(k+2))}{\sin(\pi/(k+2))}, \tag{3.9}$$

where $a = 2j$ with $j = 0, 1/2, \ldots, k/2$ is the spin of the representation. The only nontrivial case with quantum dimension smaller than two is related to $a = 1$, the spin-1/2, which has quantum dimension $d_1 = 2\cos\left(\frac{\pi}{k+2}\right)$. The CLE drift is

$$\kappa = 4\frac{k+2}{k+3}, \tag{3.10}$$

the same as the result of [25] which was found by using null vector relations. We note that just for $k = 4$ we have another quantum dimension which is equal to 2 and related to $a = 2$, spin-1, the corresponding CLE drift is $\kappa = 4$. We note that this result is the same as the result coming from A_{k+1} with the same adjacency graph and so in close connection with the modular invariant partition functions of this model. The other example is $SU_k(3)$ with the quantum dimensions

$$d_{a,b} = \frac{\sin(\pi(a+1)/(k+3))\sin(\pi(b+1)/(k+3))\sin(\pi(a+b+2)/(k+3))}{\sin(\pi/(k+3))\sin(\pi/(k+3))\sin(2\pi/(k+3))}, \tag{3.11}$$

where $a, b \geqslant 0$ and $a + b \leqslant k$. For the simplest case $k = 1$ we have two quantum dimensions $d_{1,0} = 1$ and $d_{0,1} = 1$, for $k = 2$ we have three different quantum dimensions $d_{1,0} = d_{0,1} = d_{1,1} = 2\cos\left(\frac{\pi}{5}\right)$, $d_{2,0} = d_{0,2} = 1$. It seems that there is not a nice compact form for the general case but the calculation for larger ks is straightforward and shows that most of the quantum dimensions are greater than two, except some operators with quantum dimensions equal to one. The same calculation for the $SU_k(N)$ is tractable and for $k = 2$ one of the critical loop models has the following CLE drift:

$$\kappa = 4\frac{N+2}{N+3}. \tag{3.12}$$

This is similar to equation (3.10) if we replace k with N; this is a general property in $SU_k(N)$ models and a reminiscent of the level-rank duality. One can find the quantum dimension of the most general affine Lie algebras as the weights of loop models and try to find the CLE drift by using $O(n)$ results; to find quantum dimensions of many CFTs, see [5].

3.3. Coset models

$Z(k)$ model: The same calculation can be done for the $Z(k)$ models related to the coset, $\frac{SU_k(2)}{U(1)}$, with the same fusion rules and operator content but with the different conformal weights. The central charge of this model is $c = \frac{2(k-1)}{k+2}$ and the conformal weights of the primary fields $\phi_{l,m}$ are given by $\Delta_{l,m} = \frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k}$ with $0 \leqslant l \leqslant k$, $0 \leqslant |m| \leqslant 2k-1$ and $l - m \in 2\mathbf{Z}$ so that we should just take half of the grid to remove the redundancy, the spin operator is corresponding to $m = l$ in this notation. The modular S matrix is given explicitly by

$$S_{i,j}^{m,n} = \frac{1}{\sqrt{k(k+2)}}\sin\frac{\pi(i+1)(j+1)}{k+2}\exp\left(\mathrm{i}\frac{\pi jn}{k}\right). \tag{3.13}$$

Since these models have the same fusion rules as the $SU_k(2)$ models we find the same quantum dimensions and so re-derive equation (3.10) for the first spin operator of this model. For this case if we go to the dense phase of the loop model then we have

$$\kappa = 4\frac{k+2}{k+1}, \tag{3.14}$$

which is the same as the formula proposed in [26] for the lattice $Z(k)$ models in the FK representation. So far, we were not able to find the formula for the spin representation of these models by the familiar S matrix of $Z(k)$ models. The $Z(2)$ case is the Ising model and is equal to the minimal model $M(4, 3)$ and so there is not any ambiguity in this case. For $Z(3)$ it is true that the model is equal to the three-state Potts model in the spin representation, however the operator content of this model is not equal to $M(6, 5)$. However, the two disordered operators appearing in the Dihedral description of the three-state Potts model is missing in this description but equation (3.10) is giving a true answer for $k = 3$. For higher ks the result is different from the result of [26], we will come back again to this problem when we shall discuss loop models on square lattice. The above argument means that we should be careful in extending our results to the loop models in the lattice $Z(k)$ model defined by spin variables.

$\frac{SU_k(2)\oplus SU_l(2)}{SU_{k+l}(2)}$: The similarity of the quantum dimension of $\frac{SU_k(2)}{U(1)}$ and $SU_k(2)$ comes from the general property of coset models, however coset models can have the larger operator content with different conformal weights but they have most of the quantum dimensions of original affine algebra. For example, take the coset, $\frac{SU_k(2)\oplus SU_l(2)}{SU_{k+l}(2)}$, for $l = 1$ it is equal to $M(p+1, p)$ minimal models with $\kappa = 4\frac{k+2}{k+3}$ for operator (1, 2) which is equal to the $SU_k(2)$ case, however the $M(p+1, p)$ minimal models have other critical loop representations as well. Generalization of these results to more complicated coset models is straightforward and we discussed some of them in the appendix.

4. Loop models on the square lattice

The definition of loop models on the square lattice is similar to the honeycomb lattice. From some points of view it is more natural, for example for the honeycomb lattice we introduced an unknown parameter x which should be fine-tuned to get a critical loop model, but on the square lattice just the S matrix is enough to get a critical loop model. The definition is as follows: put some heights on the vertices of the lattice so that the neighbor sites be adjacent on the fusion matrix graph. As for the honeycomb lattice, if the fusion graph has no cycles smaller than five then with probability one, one of the diagonally opposite pairs of heights in each elementary plaquette should be equal, we will use this constraint for the definition of loops. Labeling the heights by (a, b, c, d), the weight around the elementary plaquette is

$$W(a, b, c, d) = \left(\frac{S_b S_d}{S_a S_c}\right)^{\frac{1}{4}} \delta_{ac} + \lambda \left(\frac{S_a S_c}{S_b S_d}\right)^{\frac{1}{4}} \delta_{bd}, \tag{4.1}$$

where $\lambda = 1$ for the unitary minimal models. So far the definition is similar to the non-dilute ADE models but as we noted in the first section there are some fusion matrices with diagonal terms. This means that it is possible to have some neighboring vertices with the same heights, but still we will have at least a diagonally opposite pair of heights in each elementary plaquette with the same height for most of the cases that we are interested. For these cases we define the weight of the plaquette by using the loop model as follows: in each elementary plaquette we draw an edge connecting the diagonal sites if they are equal then we will have a Fortuin–Kasteleyn-like graph [18] and then the definition of the loop model on the medial lattice is straightforward just as Pott's model, figure 4. Each part of the loop inside the plaquette carries a weight $\left(\frac{S_b}{S_a}\right)^{\frac{1}{4}}$. For the plaquette with three equal heights for drawing the edges we do not have any ambiguity and we can use equation (4.1) to get the weight of the plaquette. For the plaquette with four equal heights we chose one of the edges like percolation with weights equal to one and after that the definition of the loop model is straightforward. If we consider the weights of all parts of the loop and then sum over all values of the heights consistent with a configuration of loops we will have an $O(n)$ model with $x = 1$ which is exactly the same as the loops in the FK representation of the $Q = d_a^2$ state Potts model. Following [31] there should be some holomorphic operators with conformal weights equal to s satisfying the equation $d_a = 2\sin\left(s\frac{\pi}{2}\right)$. This holomorphic operators describe the probability that the curve

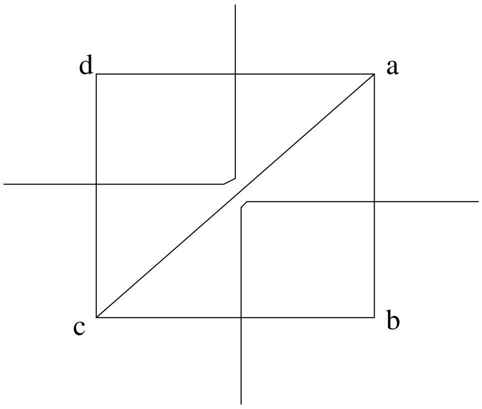

Figure 4. A plaquette of square lattice with $a = c$ and the corresponding curve segment on the dual square lattice.

passes between two close points. Then the corresponding drift term of CLE is

$$\kappa = \frac{8}{1+s}. \tag{4.2}$$

The above argument shows that we can map all of the loop models defined by the above method to the Potts model and then by using the known results of the critical Potts model we can get all of the properties of loop models. Let us investigate some examples.

4.1. Example 1: minimal models

For the case $(r, s) = (2, 1)$ it is easy to see that the spin of corresponding operator should be equal to $h_{3,1} = 2\frac{p}{p'} - 1$ and we have $\kappa = 4\frac{p'}{p}$ consistent with the argument in [31]. One can find the results for the other cases by using the same techniques. In all of the cases the results are consistent with the results of the previous section.

4.2. Example 2: $Z(k)$ model

The calculation for the parafermionic models are the same and we can easily find the connection of these CFTs to CLE by the above method. One of the famous examples is the $Z(k)$ model with the following connection to CLE for the operators with spin-$\frac{1}{2}$ representation,

$$\kappa = 4\frac{k+2}{k+1}, \tag{4.3}$$

where the corresponding holomorphic operator describing the probability that the curve passes between two close points is: $s = \frac{k}{k+2}$. We conjecture that the above formula, as Santachiara pointed out, could correspond to the FK representation of the lattice $Z(k)$ model with $k > 3$. However, the result is coming again from the different operator; in [26] the author found that the above equation, by using one of the disordered operators, comes from dihedral group but we do not have these operators in our calculations. This is another example for different primary operators with the same quantum dimensions. We will face this ambiguity again when we discuss the lattice $Z(4)$ model, Ashkin–Teller, as an orbifold model [32]. However, there are some evidences that the $Z(k)$ parafermionic model is a good candidate for the lattice $Z(k)$ model [33] but it seems that it is not enough for describing all of the properties of the model. In addition, it shows that it is important to investigate the fusion matrix of the dihedral group to improve our understanding about lattice $Z(k)$ models. For $Z(k)$ parafermionic CFT there is another more straightforward evidence that formula (4.3) is a good candidate, the argument is as follows: in equation (4.1) if we put $\lambda = -1$ then the corresponding CFT describing the height model is the $Z(k)$ model which is also the antiferromagnetic Potts model [7]; see also [34], and related to loop model with weight $-d_a$. Then again by following [31] it is easy to show that there is an antiholomorphic operator with spin $d_a = -2\sin\left(s\frac{\pi}{2}\right)$; this antiholomorphic operator is describing the probability that the curve passes between two close points but in this time by the following connection to the CLE drift:

$$\kappa = \frac{8}{1-p}, \tag{4.4}$$

which again gives equation (4.3). The tricky point is we need to work with the antiholomorphic operator in this case.

To close this section, we will just mention that the loop models coming from the definition on the square lattice are just the dense phase of loop models discussed in the previous sections; this is easy to understand if we suppose $x = 1$ in the $O(n)$ model's partition function. This is always true because for $-2 \leqslant n \leqslant 2$, x_c is always smaller than 1.

5. Discussion: connection to BCFT

In this section we would like to discuss briefly the connection of the argument of section 2 to boundary conformal field theory (BCFT) and SLE. First, let us define boundary condition for height models by following [11, 18]: for the height models on the square lattice suppose the wired boundary conditions on the real line but with height a on the left part and b on the right side. If we work on the half plane then as well as some nested loops we will have a curve going from the origin to infinity if a and b are adjacent on the corresponding graph, if they do not then it is possible to have more than one curve. Let us first suppose the adjacent case then we will have a curve and following [31] it is possible to relate the probability that the curve passes between two closed points to a holomorphic operator with spin s, mentioned in the previous section, with the expectation value $\langle\psi(z)\rangle \approx \frac{1}{z^p}$. This argument could be the starting point for a rigorous proof of the connection of these curves to SLE as Smirnov used it to prove the convergence in the Ising case [35]. The above definition is possible for all of the examples that we discussed in section 3 independent of the null vector property of the corresponding primary operator. This is paradoxical because we know from the work of Bauer and Bernard [16], and Friedrich and Werner[36] that the existence of null vectors at level 2 is essential to get SLE. The generalized stochastic equations of [25, 26] showed that the extra symmetries in the theory can modify the level 2 null vector to more complicated null vector, so the Virasoro null vector is not complete for describing theories with extra symmetries. This can be one of the reasons that we can get a large set of loop models for different CFTs without concerning the null vector properties of operators, however the exact connection is missing so far.

The other important point in the connection of our loop models to BCFT is: why are we just working with bulk operators without speaking about boundary states? This comes from the old result of Cardy [37] stating that there is a bijection between the possible conformally invariant boundary conditions and the bulk primary operators. Let us see this more carefully: in the upper half plane for the above boundary condition it was shown in [37] that only one copy of the chiral algebra acts and so it is possible to write the partition function on the cylinder as a sum on the characters of chiral parts

$$Z_{ba} = \sum_i n_{ib}^a \chi_i(\tau), \qquad (5.1)$$

where the sum is over all of the primary operators and n_{ib}^a is the element of the matrix n_i satisfying the Verlinde fusion algebra [38] with the eigenvalues the same as for the fusion matrix. So one can draw the graph corresponding to n_i and find the same loop representation as we found. This is the method followed by [38] to classify all the conformal boundary conditions of rational CFTs, specially $SU(2)$ WZW models. The above argument teaches us that the discussed loop models are compatible with conformal boundary conditions and with SLE, however it seems that this statement is highly nontrivial and needs more investigation [39]. For example, we used Cardy's equation as a consistency condition for conformal boundary conditions but it is not trivial why in the lattice model when we go to the continuum the model is conformally invariant. This comes from an old problem in statistical physics at the critical point is the statistical mechanics models at this point conformally invariant or invariant at least at the critical loop levels? There is much evidence that the answer is yes for most of the exactly solved models but SLE is the only method so far giving us a mechanism for a rigorous proof.

6. Conclusions

We showed that it is possible to define a height model for a generic primary operator of general CFT by using the fusion matrix as an adjacency matrix. In addition, it is possible to associate with these height models some loop models with loop weights equal to the quantum dimension of the corresponding primary operator.

In the critical regimes, these new loop models have some properties similar to those of the loops corresponding to CFT. For example, the loop model corresponding to $M(3, 4)$ has the same properties as the Ising model's loop representation. In the lattice level it is not completely clear why this should happen but it gives us a good mechanism to find the loop properties of lattice models by studying the corresponding CFT without going to lattice level and studying the properties of discrete variables which most of the time is quite difficult. Investigating the connection with the integrable ADE models is enlightening in better understanding the connection with lattice models. Of course, finding the CFT of the lattice model is not simple so we think that our method can be useful for the more studied models with the known operator content. In this paper, we just list some simple CFTs but as we mentioned before it is possible to investigate the loop models corresponding to the more general CFTs from affine Lie algebras to supersymmetric models and coset models. It is also interesting to generalize the height models to the fused cases and find the connection to the loop models with two kinds of loops [7]. The other more interesting direction is using the method of Behrend *et al* [38] to find the fusion rules of boundary conformal field theory by just using the modular invariant partition function of the CFT and then finding the loop model representation by the method that we described in [39].

Moreover, we think that this method of generation of loop models can be useful to classify the ground state of topological quantum theories because as we already mentioned before the weights of the above loop models are exactly the same as quantum dimensions of the operators appearing in the CFT. The primary operators of CFT describe the edge states of topological theory and also some statistical properties of the model, specially the topological quantum entanglement [40, 41].

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Appendix A. Loop models for more general coset theories

In this appendix, we are going to summarize some of the possible loop models of more general coset theories.

$\frac{SU_k(2)\oplus SU_l(2)}{SU_{k+l}(2)}$: The case $l = 1$ was already discussed in section 3. The case $l = 2$ is related to $N = 1$ superconformal minimal models and should have critical loops related to the loops of $SU_k(2)$. To get all of the critical loop representations of $N = 1$ superconformal models one should investigate the S matrix of these models carefully. Let us see this case with more detail. Take $k = m - 2$ then the central charge of this model is $c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right)$ for $m = 3, 4, 5, \ldots$ and one can label the conformal weights by the triplets (j, k, l) with $k = 0, 1, \ldots, m$, $j = 0, 1, \ldots, m - 2$ and $l = 0, 1, 2$. We just consider the triplets with $j - k + l$ being even and the symmetry $(j, k, l) = (m - 2 - j, m - k, 2 - l)$, so by using the S matrix given in [27, 28] we will have the following quantum dimensions:

$$d_{(j,k,l)} = \frac{\sin\left(\pi\frac{j+1}{m}\right)\sin\left(\pi\frac{k+1}{m+2}\right)\sin\left(\pi\frac{l+1}{4}\right)}{\sin\left(\frac{\pi}{m+2}\right)\sin\left(\frac{\pi}{m}\right)\sin\left(\frac{\pi}{4}\right)}. \tag{A.1}$$

For $m = 3$ this theory is equal to the $M(5, 4)$ minimal model describing three critical Ising models and one can find the same results as example 1. For $m = 4$ one can find the following conformal dimensions:

$$\begin{aligned}
&d_{(0,0,0)} = 1, && d_{(0,2,0)} = 2, && d_{(0,4,0)} = 1, && d_{(1,1,0)} = \sqrt{6},\\
&d_{(1,3,0)} = \sqrt{6}, && d_{(2,0,0)} = 1, && d_{(2,2,0)} = 2, && d_{(2,4,0)} = 1,\\
&d_{(0,3,1)} = \sqrt{6}, && d_{(1,4,1)} = 2, && d_{(2,3,1)} = \sqrt{6}, && d_{(1,2,1)} = 2\sqrt{3}.
\end{aligned} \tag{A.2}$$

Using equation (3.2) one can find the drift of CLE easily. The extension to general m shows that we have a primary operator $(0, m - 1, 0)$ for odd m with the graph related to A_{m+1} and so with the following CLE drifts

$$\kappa = 4\frac{m+2}{m+3}, \qquad \kappa = 4\frac{m+2}{m+1}, \tag{A.3}$$

for dilute and dense cases, respectively. If we drop the constraint on the triplets then we can find the same equation for even m if we chose $(0, m - 1, 0)$ which is indeed possible if we work with the modified partition function [28]. It is not difficult to show that for $(m - 3, 0, 0)$ the corresponding graph is A_{m-1}. These loop models can be related to [6] which gives a lattice loop candidate for supersymmetric models.

Finally, we conjecture that for the more general case, $\frac{SU_k(2)\oplus SU_l(2)}{SU_{k+l}(2)}$, we should have at least the following conformal curves:

$$\kappa = 4\frac{k+2}{k+3}, \qquad \kappa = 4\frac{l+2}{l+3}, \qquad \kappa = 4\frac{k+l+2}{k+l+3}. \tag{A.4}$$

We think that these are related to critical loops of fused RSOS models introduced in [29]. The important thing to mention here is that most of the loop models that we can extract with our method in this case are non-critical and cannot be consistent. There is a method for resolving this by using the coupled loop models introduced recently by Fendley[3] [6] which is based on decoupling the quantum dimension to the product of generally two different loop weights for fused graphs.

$\frac{SU_{k-1}(N)\oplus SU_1(N)}{SU_k(N)}$: These coset models are called W_N algebra and describing the $\mathbf{A}^n$ critical models [30], the central charge of these models is $c = (N-1)\big(1 - \frac{N(N+1)}{(k+N)(k+N-1)}\big)$. For these models by getting motivation from the quantum dimensions that we calculated for the $SU_k(N)$ one can show that for every $N > 2$ we have just critical loop models with the following CLE drift for $k = 2$:

$$\kappa = 4\frac{N+2}{N+3}. \tag{A.5}$$

It is not obvious why the $k = 2$ case is an exception here but this formalism is showing that for $k > 2$ we cannot find a nontrivial critical loop model. It seems that there should also be the same decoupling procedure as [6], however this case is totally unexplored. The above formula gives the true answer for $N = 3$ and $k = 2$ which is related to the three-state Potts model. It is not difficult to see that $k = 2$ is related to the $Z(N)$ model and the above equation is in fact the same as equation (3.14) that we found for the simplest parafermionic model.

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